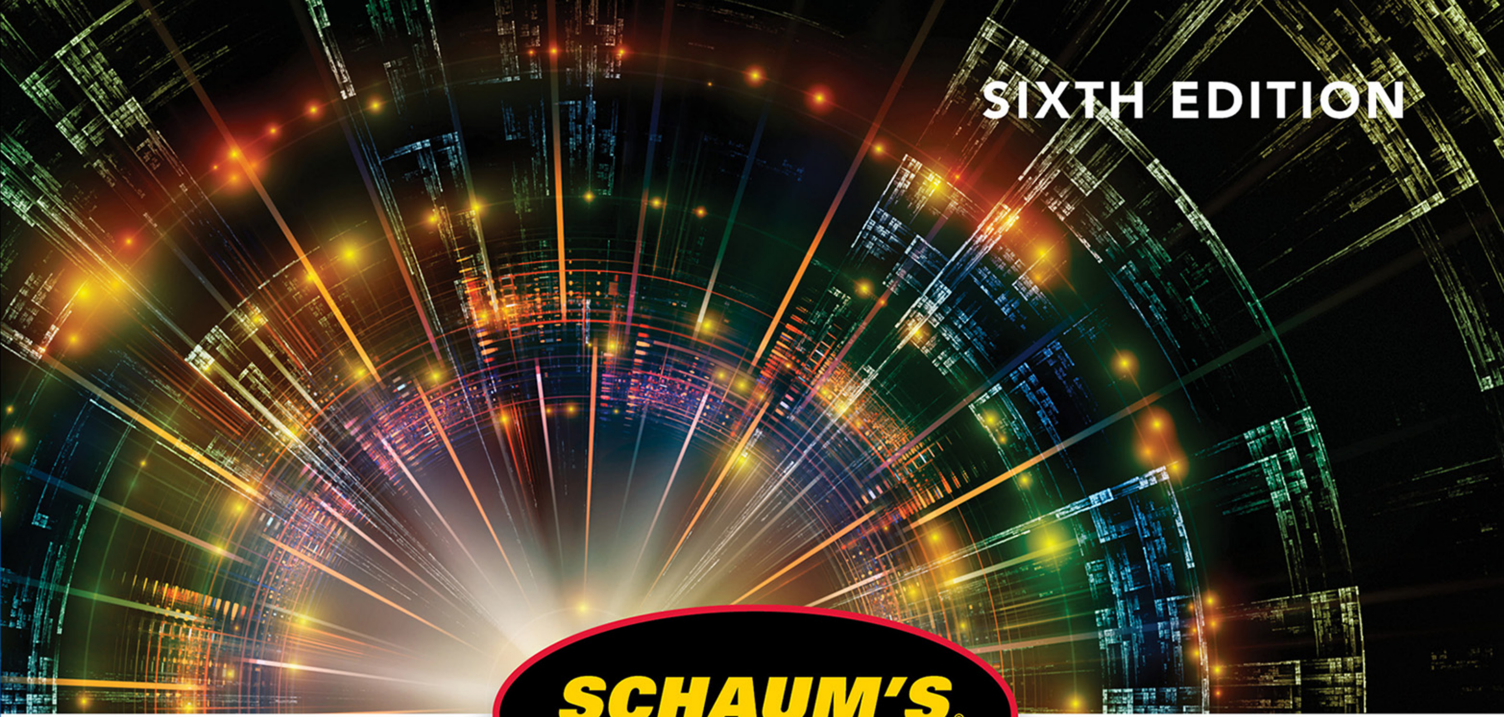

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Sixth Edition

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Preface

Linear algebra has in recent years become an essential part of the mathematical background required by mathematicians and mathematics teachers, engineers, computer scientists, physicists, economists, and statisticians, among others. This requirement reflects the importance and wide applications of the subject matter.

This book is designed for use as a textbook for a formal course in linear algebra or as a supplement to all current standard texts. It aims to present an introduction to linear algebra which will be found helpful to all readers regardless of their fields of specification. More material has been included than can be covered in most first courses. This has been done to make the book more flexible, to provide a useful book of reference, and to stimulate further interest in the subject.

Each chapter begins with clear statements of pertinent definitions, principles, and theorems together with illustrative and other descriptive material. This is followed by graded sets of solved and supplementary problems. The solved problems serve to illustrate and amplify the theory, and to provide the repetition of basic principles so vital to effective learning. Numerous proofs, especially those of all essential theorems, are included among the solved problems. The supplementary problems serve as a complete review of the material of each chapter.

The first three chapters treat vectors in Euclidean space, matrix algebra, and systems of linear equations. These chapters provide the motivation and basic computational tools for the abstract investigations of vector spaces and linear mappings which follow. After chapters on inner product spaces and orthogonality and on determinants, there is a detailed discussion of eigenvalues and eigenvectors giving conditions for representing a linear operator by a diagonal matrix. This naturally leads to the study of various canonical forms, specifically, the triangular, Jordan, and rational canonical forms. Later chapters cover linear functions and the dual space V^*, and bilinear, quadratic, and Hermitian forms. The last chapter treats linear operators on inner product spaces.

The main changes in the sixth edition are that some parts in Appendix D have been added to the main part of the text, that is, Chapter Four and Chapter Eight. There are also many additional solved and supplementary problems.

Finally, we wish to thank the staff of the McGraw-Hill Schaum's Outline Series, especially Diane Grayson, for their unfailing cooperation.

SEYMOUR LIPSCHUTZ
MARC LARS LIPSON

List of Symbols

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CHAPTER 1

Vectors in R^n and C^n, Spatial Vectors

1.1 Introduction

There are two ways to motivate the notion of a vector: one is by means of lists of numbers and subscripts, and the other is by means of certain objects in physics. We discuss these two ways below.

Here we assume the reader is familiar with the elementary properties of the field of real numbers, denoted by **R**. On the other hand, we will review properties of the field of complex numbers, denoted by **C**. In the context of vectors, the elements of our number fields are called *scalars*.

Although we will restrict ourselves in this chapter to vectors whose elements come from **R** and then from **C**, many of our operations also apply to vectors whose entries come from some arbitrary field K.

Lists of Numbers

Suppose the weights (in pounds) of eight students are listed as follows:

$$156, \quad 125, \quad 145, \quad 134, \quad 178, \quad 145, \quad 162, \quad 193$$

One can denote all the values in the list using only one symbol, say w, but with different subscripts; that is,

$$w_1, \quad w_2, \quad w_3, \quad w_4, \quad w_5, \quad w_6, \quad w_7, \quad w_8$$

Observe that each subscript denotes the position of the value in the list. For example,

$$w_1 = 156, \text{ the first number}, \; w_2 = 125, \text{ the second number}, \; \ldots$$

Such a list of values,

$$w = (w_1, w_2, w_3, \ldots, w_8)$$

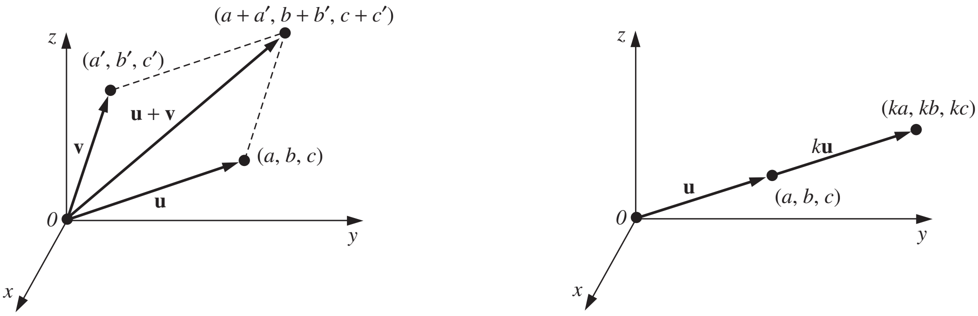
is called a *linear array* or *vector*.

Vectors in Physics

Many physical quantities, such as temperature and speed, possess only "magnitude." These quantities can be represented by real numbers and are called *scalars*. On the other hand, there are also quantities, such as force and velocity, that possess both "magnitude" and "direction." These quantities, which can be represented by arrows having appropriate lengths and directions and emanating from some given reference point O, are called *vectors*.

Now we assume the reader is familiar with the space $\mathbf{R}^3$ where all the points in space are represented by ordered triples of real numbers. Suppose the origin of the axes in $\mathbf{R}^3$ is chosen as the reference point O for the vectors discussed above. Then every vector is uniquely determined by the coordinates of its endpoint, and vice versa.

There are two important operations, vector addition and scalar multiplication, associated with vectors in physics. The definition of these operations and the relationship between these operations and the endpoints of the vectors are as follows.



(*a*) Vector Addition　　(*b*) Scalar Multiplication

Figure 1-1

(i) ***Vector Addition:*** The resultant $\mathbf{u}+\mathbf{v}$ of two vectors $\mathbf{u}$ and $\mathbf{v}$ is obtained by the *parallelogram law*; that is, $\mathbf{u}+\mathbf{v}$ is the diagonal of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$. Furthermore, if (a,b,c) and (a',b',c') are the endpoints of the vectors $\mathbf{u}$ and $\mathbf{v}$, then $(a+a',\ b+b',\ c+c')$ is the endpoint of the vector $\mathbf{u}+\mathbf{v}$. These properties are pictured in Fig. 1-1(a).

(ii) ***Scalar Multiplication:*** The product $k\mathbf{u}$ of a vector $\mathbf{u}$ by a real number k is obtained by multiplying the magnitude of $\mathbf{u}$ by k and retaining the same direction if $k>0$ or the opposite direction if $k<0$. Also, if (a,b,c) is the endpoint of the vector $\mathbf{u}$, then (ka,kb,kc) is the endpoint of the vector $k\mathbf{u}$. These properties are pictured in Fig. 1-1(b).

Mathematically, we identify the vector $\mathbf{u}$ with its (a,b,c) and write $\mathbf{u}=(a,b,c)$. Moreover, we call the ordered triple (a,b,c) of real numbers a point or vector depending upon its interpretation. We generalize this notion and call an n-tuple $(a_1,a_2,\ldots,a_n)$ of real numbers a vector. However, special notation may be used for the vectors in $\mathbf{R}^3$ called *spatial* vectors (Section 1.6).

1.2 Vectors in $\mathbf{R}^n$

The set of all n-tuples of real numbers, denoted by $\mathbf{R}^n$, is called *n-space*. A particular n-tuple in $\mathbf{R}^n$, say

$$u=(a_1,a_2,\ldots,a_n)$$

is called a *point* or *vector*. The numbers a_i are called the *coordinates*, *components*, *entries*, or *elements* of u. Moreover, when discussing the space $\mathbf{R}^n$, we use the term *scalar* for the elements of $\mathbf{R}$.

Two vectors, u and v, are *equal*, written $u=v$, if they have the same number of components and if the corresponding components are equal. Although the vectors $(1,2,3)$ and $(2,3,1)$ contain the same three numbers, these vectors are not equal because corresponding entries are not equal.

The vector $(0,0,\ldots,0)$ whose entries are all 0 is called the *zero vector* and is usually denoted by 0.

EXAMPLE 1.1

(a) The following are vectors:

$$(2,-5),\quad (7,9),\quad (0,0,0),\quad (3,4,5)$$

The first two vectors belong to $\mathbf{R}^2$, whereas the last two belong to $\mathbf{R}^3$. The third is the zero vector in $\mathbf{R}^3$.

(b) Find x,y,z such that $(x-y,\ x+y,\ z-1)=(4,2,3)$.

By definition of equality of vectors, corresponding entries must be equal. Thus,

$$x-y=4,\qquad x+y=2,\qquad z-1=3$$

Solving the above system of equations yields $x=3$, $y=-1$, $z=4$.

Column Vectors

Sometimes a vector in n-space $\mathbf{R}^n$ is written vertically rather than horizontally. Such a vector is called a *column vector*, and, in this context, the horizontally written vectors in Example 1.1 are called *row vectors*. For example, the following are column vectors with $2, 2, 3$, and 3 components, respectively:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 5 \\ -6 \end{bmatrix}, \quad \begin{bmatrix} 1.5 \\ \frac{2}{3} \\ -15 \end{bmatrix}$$

We also note that any operation defined for row vectors is defined analogously for column vectors.

1.3 Vector Addition and Scalar Multiplication

Consider two vectors u and v in $\mathbf{R}^n$, say

$$u = (a_1, a_2, \ldots, a_n) \qquad \text{and} \qquad v = (b_1, b_2, \ldots, b_n)$$

Their *sum*, written $u + v$, is the vector obtained by adding corresponding components from u and v. That is,

$$u + v = (a_1 + b_1, \ a_2 + b_2, \ \ldots, \ a_n + b_n)$$

The *product*, of the vector u by a real number k, written ku, is the vector obtained by multiplying each component of u by k. That is,

$$ku = k(a_1, a_2, \ldots, a_n) = (ka_1, ka_2, \ldots, ka_n)$$

Observe that $u + v$ and ku are also vectors in $\mathbf{R}^n$. The sum of vectors with different numbers of components is not defined.

Negatives and subtraction are defined in $\mathbf{R}^n$ as follows:

$$-u = (-1)u \qquad \text{and} \qquad u - v = u + (-v)$$

The vector $-u$ is called the *negative* of u, and $u - v$ is called the *difference* of u and v.

Now suppose we are given vectors $u_1, u_2, \ldots, u_m$ in $\mathbf{R}^n$ and scalars $k_1, k_2, \ldots, k_m$ in $\mathbf{R}$. We can multiply the vectors by the corresponding scalars and then add the resultant scalar products to form the vector

$$v = k_1u_1 + k_2u_2 + k_3u_3 + \cdots + k_mu_m$$

Such a vector v is called a *linear combination* of the vectors $u_1, u_2, \ldots, u_m$.

EXAMPLE 1.2

(a) Let $u = (2, 4, -5)$ and $v = (1, -6, 9)$. Then

$$\begin{aligned} u + v &= (2 + 1, \ 4 + (-6), \ -5 + 9) = (3, -2, 4) \\ 7u &= (7(2), 7(4), 7(-5)) = (14, 28, -35) \\ -v &= (-1)(1, -6, 9) = (-1, 6, -9) \\ 3u - 5v &= (6, 12, -15) + (-5, 30, -45) = (1, 42, -60) \end{aligned}$$

(b) The zero vector $0 = (0, 0, \ldots, 0)$ in $\mathbf{R}^n$ is similar to the scalar 0 in that, for any vector $u = (a_1, a_2, \ldots, a_n)$.

$$u + 0 = (a_1 + 0, \ a_2 + 0, \ \ldots, \ a_n + 0) = (a_1, a_2, \ldots, a_n) = u$$

(c) Let $u = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$. Then $2u - 3v = \begin{bmatrix} 4 \\ 6 \\ -8 \end{bmatrix} + \begin{bmatrix} -9 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -5 \\ 9 \\ -2 \end{bmatrix}$.

Basic properties of vectors under the operations of vector addition and scalar multiplication are described in the following theorem.

THEOREM 1.1: For any vectors u, v, w in $\mathbf{R}^n$ and any scalars k, k' in $\mathbf{R}$,

(i) $(u+v)+w = u+(v+w)$, (v) $k(u+v) = ku+kv$,

(ii) $u+0 = u$, (vi) $(k+k')u = ku+k'u$,

(iii) $u+(-u) = 0$, (vii) $(kk')u = k(k'u)$,

(iv) $u+v = v+u$, (viii) $1u = u$.

We postpone the proof of Theorem 1.1 until Chapter 2, where it appears in the context of matrices (Problem 2.3).

Suppose u and v are vectors in $\mathbf{R}^n$ for which $u = kv$ for some nonzero scalar k in $\mathbf{R}$. Then u is called a *multiple* of v. Also, u is said to be in the *same* or *opposite direction* as v according to whether $k > 0$ or $k < 0$.

1.4 Dot (Inner) Product

Consider arbitrary vectors u and v in $\mathbf{R}^n$; say,

$$u = (a_1, a_2, \ldots, a_n) \qquad \text{and} \qquad v = (b_1, b_2, \ldots, b_n)$$

The *dot product* or *inner product* of u and v is denoted and defined by

$$u \cdot v = a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

That is, $u \cdot v$ is obtained by multiplying corresponding components and adding the resulting products. The vectors u and v are said to be *orthogonal* (or *perpendicular*) if their dot product is zero—that is, if $u \cdot v = 0$.

EXAMPLE 1.3

(a) Let $u = (1, -2, 3)$, $v = (4, 5, -1)$, $w = (2, 7, 4)$. Then,

$$u \cdot v = 1(4) - 2(5) + 3(-1) = 4 - 10 - 3 = -9$$
$$u \cdot w = 2 - 14 + 12 = 0, \qquad v \cdot w = 8 + 35 - 4 = 39$$

Thus, u and w are orthogonal.

(b) Let $u = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$. Then $u \cdot v = 6 - 3 + 8 = 11$.

(c) Suppose $u = (1, 2, 3, 4)$ and $v = (6, k, -8, 2)$. Find k so that u and v are orthogonal.

First obtain $u \cdot v = 6 + 2k - 24 + 8 = -10 + 2k$. Then set $u \cdot v = 0$ and solve for k:

$$-10 + 2k = 0 \qquad \text{or} \qquad 2k = 10 \qquad \text{or} \qquad k = 5$$

Basic properties of the dot product in $\mathbf{R}^n$ (proved in Problem 1.13) follow.

THEOREM 1.2: For any vectors u, v, w in $\mathbf{R}^n$ and any scalar k in $\mathbf{R}$:

(i) $(u+v) \cdot w = u \cdot w + v \cdot w$, (iii) $u \cdot v = v \cdot u$,

(ii) $(ku) \cdot v = k(u \cdot v)$, (iv) $u \cdot u \geq 0$, and $u \cdot u = 0$ iff $u = 0$.

Note that (ii) says that we can "take k out" from the first position in an inner product. By (iii) and (ii),

$$u \cdot (kv) = (kv) \cdot u = k(v \cdot u) = k(u \cdot v)$$

That is, we can also "take k out" from the second position in an inner product.

The space $\mathbf{R}^n$ with the above operations of vector addition, scalar multiplication, and dot product is usually called *Euclidean n-space*.

Norm (Length) of a Vector

The *norm* or *length* of a vector u in $\mathbf{R}^n$, denoted by $\|u\|$, is defined to be the nonnegative square root of $u \cdot u$. In particular, if $u = (a_1, a_2, \ldots, a_n)$, then

$$\|u\| = \sqrt{u \cdot u} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

That is, $\|u\|$ is the square root of the sum of the squares of the components of u. Thus, $\|u\| \geq 0$, and $\|u\| = 0$ if and only if $u = 0$.

A vector u is called a *unit* vector if $\|u\| = 1$ or, equivalently, if $u \cdot u = 1$. For any nonzero vector v in $\mathbf{R}^n$, the vector

$$\hat{v} = \frac{1}{\|v\|} v = \frac{v}{\|v\|}$$

is the unique unit vector in the same direction as v. The process of finding $\hat{v}$ from v is called *normalizing* v.

EXAMPLE 1.4

(a) Suppose $u = (1, -2, -4, 5, 3)$. To find $\|u\|$, we can first find $\|u\|^2 = u \cdot u$ by squaring each component of u and adding, as follows:

$$\|u\|^2 = 1^2 + (-2)^2 + (-4)^2 + 5^2 + 3^2 = 1 + 4 + 16 + 25 + 9 = 55$$

Then $\|u\| = \sqrt{55}$.

(b) Let $v = (1, -3, 4, 2)$ and $w = (\frac{1}{2}, -\frac{1}{6}, \frac{5}{6}, \frac{1}{6})$. Then

$$\|v\| = \sqrt{1 + 9 + 16 + 4} = \sqrt{30} \qquad \text{and} \qquad \|w\| = \sqrt{\frac{9}{36} + \frac{1}{36} + \frac{25}{36} + \frac{1}{36}} = \sqrt{\frac{36}{36}} = \sqrt{1} = 1$$

Thus w is a unit vector, but v is not a unit vector. However, we can normalize v as follows:

$$\hat{v} = \frac{v}{\|v\|} = \left(\frac{1}{\sqrt{30}}, \frac{-3}{\sqrt{30}}, \frac{4}{\sqrt{30}}, \frac{2}{\sqrt{30}}\right)$$

This is the unique unit vector in the same direction as v.

The following formula (proved in Problem 1.14) is known as the Schwarz inequality or Cauchy–Schwarz inequality. It is used in many branches of mathematics.

THEOREM 1.3 (Schwarz): For any vectors u, v in $\mathbf{R}^n$, $|u \cdot v| \leq \|u\|\|v\|$.

Using the above inequality, we also prove (Problem 1.15) the following result known as the "triangle inequality" or Minkowski's inequality.

THEOREM 1.4 (Minkowski): For any vectors u, v in $\mathbf{R}^n$, $\|u + v\| \leq \|u\| + \|v\|$.

Distance, Angles, Projections

The *distance* between vectors $u = (a_1, a_2, \ldots, a_n)$ and $v = (b_1, b_2, \ldots, b_n)$ in $\mathbf{R}^n$ is denoted and defined by

$$d(u, v) = \|u - v\| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2}$$

One can show that this definition agrees with the usual notion of distance in the Euclidean plane $\mathbf{R}^2$ or space $\mathbf{R}^3$.

The *angle* θ between nonzero vectors u, v in $\mathbf{R}^n$ is defined by

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|}$$

This definition is well defined, because, by the Schwarz inequality (Theorem 1.3),

$$-1 \le \frac{u \cdot v}{\|u\|\|v\|} \le 1$$

Note that if $u \cdot v = 0$, then $\theta = 90^\circ$ (or $\theta = \pi/2$). This then agrees with our previous definition of orthogonality.

The *projection* of a vector u onto a nonzero vector v is the vector denoted and defined by

$$\text{proj}(u, v) = \frac{u \cdot v}{\|v\|^2} v = \frac{u \cdot v}{v \cdot v} v$$

We show below that this agrees with the usual notion of vector projection in physics.

EXAMPLE 1.5

(a) Suppose $u = (1, -2, 3)$ and $v = (2, 4, 5)$. Then

$$d(u, v) = \sqrt{(1-2)^2 + (-2-4)^2 + (3-5)^2} = \sqrt{1 + 36 + 4} = \sqrt{41}$$

To find $\cos\theta$, where θ is the angle between u and v, we first find

$$u \cdot v = 2 - 8 + 15 = 9, \qquad \|u\|^2 = 1 + 4 + 9 = 14, \qquad \|v\|^2 = 4 + 16 + 25 = 45$$

Then

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|} = \frac{9}{\sqrt{14}\sqrt{45}}$$

Also,

$$\text{proj}(u, v) = \frac{u \cdot v}{\|v\|^2} v = \frac{9}{45}(2, 4, 5) = \frac{1}{5}(2, 4, 5) = \left(\frac{2}{5}, \frac{4}{5}, 1\right)$$

(b) Consider the vectors u and v in Fig. 1-2(a) (with respective endpoints A and B). The (perpendicular) projection of u onto v is the vector u^* with magnitude

$$\|u^*\| = \|u\| \cos\theta = \|u\| \frac{u \cdot v}{\|u\|\|v\|} = \frac{u \cdot v}{\|v\|}$$

To obtain u^*, we multiply its magnitude by the unit vector in the direction of v, obtaining

$$u^* = \|u^*\| \frac{v}{\|v\|} = \frac{u \cdot v}{\|v\|} \frac{v}{\|v\|} = \frac{u \cdot v}{\|v\|^2} v$$

This is the same as the above definition of proj(u, v).

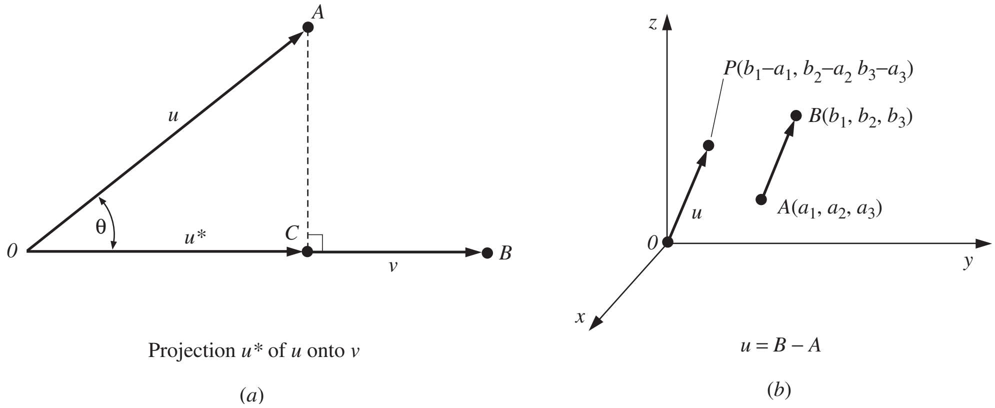


Figure 1-2

1.5 Located Vectors, Hyperplanes, Lines, Curves in R^n

This section distinguishes between an n-tuple $P(a_i) \equiv P(a_1, a_2, \ldots, a_n)$ viewed as a point in $\mathbf{R}^n$ and an n-tuple $u = [c_1, c_2, \ldots, c_n]$ viewed as a vector (arrow) from the origin O to the point $C(c_1, c_2, \ldots, c_n)$.

Located Vectors

Any pair of points $A(a_i)$ and $B(b_i)$ in $\mathbf{R}^n$ defines the *located vector* or *directed line segment* from A to B, written $\overrightarrow{AB}$. We identify $\overrightarrow{AB}$ with the vector

$$u = B - A = [b_1 - a_1,\ b_2 - a_2,\ \ldots,\ b_n - a_n]$$

because $\overrightarrow{AB}$ and u have the same magnitude and direction. This is pictured in Fig. 1-2(b) for the points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ in $\mathbf{R}^3$ and the vector $u = B - A$ which has the endpoint $P(b_1 - a_1,\ b_2 - a_2,\ b_3 - a_3)$.

Hyperplanes

A *hyperplane* H in $\mathbf{R}^n$ is the set of points $(x_1, x_2, \ldots, x_n)$ that satisfy a linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where the vector $u = [a_1, a_2, \ldots, a_n]$ of coefficients is not zero. Thus a hyperplane H in $\mathbf{R}^2$ is a line, and a hyperplane H in $\mathbf{R}^3$ is a plane. We show below, as pictured in Fig. 1-3(a) for $\mathbf{R}^3$, that u is orthogonal to any directed line segment $\overrightarrow{PQ}$, where $P(p_i)$ and $Q(q_i)$ are points in H. [For this reason, we say that u is *normal* to H and that H is *normal* to u.]

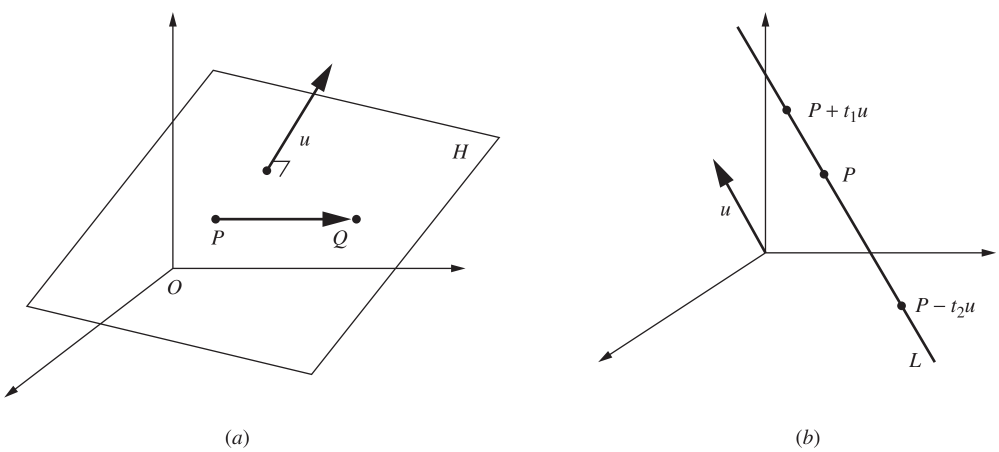


Figure 1-3

Because $P(p_i)$ and $Q(q_i)$ belong to H, they satisfy the above hyperplane equation—that is,

$$a_1p_1 + a_2p_2 + \cdots + a_np_n = b \quad \text{and} \quad a_1q_1 + a_2q_2 + \cdots + a_nq_n = b$$

Let $\quad v = \overrightarrow{PQ} = Q - P = [q_1 - p_1, q_2 - p_2, \ldots, q_n - p_n]$

Then

$$\begin{aligned} u \cdot v &= a_1(q_1 - p_1) + a_2(q_2 - p_2) + \cdots + a_n(q_n - p_n) \\ &= (a_1q_1 + a_2q_2 + \cdots + a_nq_n) - (a_1p_1 + a_2p_2 + \cdots + a_np_n) = b - b = 0 \end{aligned}$$

Thus $v = \overrightarrow{PQ}$ is orthogonal to u, as claimed.

Lines in $\mathbf{R}^n$

The *line* L in $\mathbf{R}^n$ passing through the point $P(b_1, b_2, \ldots, b_n)$ and in the direction of a nonzero vector $u = [a_1, a_2, \ldots, a_n]$ consists of the points $X(x_1, x_2, \ldots, x_n)$ that satisfy

$$X = P + tu \qquad \text{or} \qquad \begin{cases} x_1 = a_1 t + b_1 \\ x_2 = a_2 t + b_2 \\ \cdots\cdots\cdots\cdots\cdots \\ x_n = a_n t + b_n \end{cases} \qquad \text{or} \quad L(t) = (a_i t + b_i)$$

where the *parameter* t takes on all real values. Such a line L in $\mathbf{R}^3$ is pictured in Fig. 1-3(b).

EXAMPLE 1.6

(a) Let H be the plane in $\mathbf{R}^3$ corresponding to the linear equation $2x - 5y + 7z = 4$. Observe that $P(1, 1, 1)$ and $Q(5, 4, 2)$ are solutions of the equation. Thus P and Q and the directed line segment

$$v = \overrightarrow{PQ} = Q - P = [5 - 1,\ 4 - 1,\ 2 - 1] = [4, 3, 1]$$

lie on the plane H. The vector $u = [2, -5, 7]$ is normal to H, and, as expected,

$$u \cdot v = [2, -5, 7] \cdot [4, 3, 1] = 8 - 15 + 7 = 0$$

That is, u is orthogonal to v.

(b) Find an equation of the hyperplane H in $\mathbf{R}^4$ that passes through the point $P(1, 3, -4, 2)$ and is normal to the vector $u = [4, -2, 5, 6]$.

The coefficients of the unknowns of an equation of H are the components of the normal vector u; hence, the equation of H must be of the form

$$4x_1 - 2x_2 + 5x_3 + 6x_4 = k$$

Substituting P into this equation, we obtain

$$4(1) - 2(3) + 5(-4) + 6(2) = k \qquad \text{or} \qquad 4 - 6 - 20 + 12 = k \qquad \text{or} \qquad k = -10$$

Thus, $4x_1 - 2x_2 + 5x_3 + 6x_4 = -10$ is the equation of H.

(c) Find the parametric representation of the line L in $\mathbf{R}^4$ passing through the point $P(1, 2, 3, -4)$ and in the direction of $u = [5, 6, -7, 8]$. Also, find the point Q on L when $t = 1$.

Substitution in the above equation for L yields the following parametric representation:

$$x_1 = 5t + 1, \qquad x_2 = 6t + 2, \qquad x_3 = -7t + 3, \qquad x_4 = 8t - 4$$

or, equivalently,

$$L(t) = (5t + 1, 6t + 2, -7t + 3, 8t - 4)$$

Note that $t = 0$ yields the point P on L. Substitution of $t = 1$ yields the point $Q(6, 8, -4, 4)$ on L.

Curves in $\mathbf{R}^n$

Let D be an interval (finite or infinite) on the real line $\mathbf{R}$. A continuous function $F: D \to \mathbf{R}^n$ is a *curve* in $\mathbf{R}^n$. Thus, to each point $t \in D$ there is assigned the following point in $\mathbf{R}^n$:

$$F(t) = [F_1(t), F_2(t), \ldots, F_n(t)]$$

Moreover, the derivative (if it exists) of $F(t)$ yields the vector

$$V(t) = \frac{dF(t)}{dt} = \left[\frac{dF_1(t)}{dt}, \frac{dF_2(t)}{dt}, \ldots, \frac{dF_n(t)}{dt}\right]$$

which is tangent to the curve. Normalizing $V(t)$ yields

$$\mathbf{T}(t) = \frac{V(t)}{\|V(t)\|}$$

Thus, $\mathbf{T}(t)$ is the unit tangent vector to the curve. (Unit vectors with geometrical significance are often presented in bold type.)

EXAMPLE 1.7 Consider the curve $F(t) = [\sin t, \cos t, t]$ in $\mathbf{R}^3$. Taking the derivative of $F(t)$ [or each component of $F(t)$] yields

$$V(t) = [\cos t, -\sin t, 1]$$

which is a vector tangent to the curve. We normalize $V(t)$. First we obtain

$$\|V(t)\|^2 = \cos^2 t + \sin^2 t + 1 = 1 + 1 = 2$$

Then the unit tangent vection $\mathbf{T}(t)$ to the curve follows:

$$\mathbf{T}(t) = \frac{V(t)}{\|V(t)\|} = \left[\frac{\cos t}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$$

1.6 Vectors in R^3 (Spatial Vectors), ijk Notation

Vectors in $\mathbf{R}^3$, called *spatial vectors*, appear in many applications, especially in physics. In fact, a special notation is frequently used for such vectors as follows:

$\mathbf{i} = [1, 0, 0]$ denotes the unit vector in the x direction.

$\mathbf{j} = [0, 1, 0]$ denotes the unit vector in the y direction.

$\mathbf{k} = [0, 0, 1]$ denotes the unit vector in the z direction.

Then any vector $u = [a, b, c]$ in $\mathbf{R}^3$ can be expressed uniquely in the form

$$u = [a, b, c] = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Because the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors and are mutually orthogonal, we obtain the following dot products:

$$\mathbf{i}\cdot\mathbf{i} = 1, \quad \mathbf{j}\cdot\mathbf{j} = 1, \quad \mathbf{k}\cdot\mathbf{k} = 1 \qquad \text{and} \qquad \mathbf{i}\cdot\mathbf{j} = 0, \quad \mathbf{i}\cdot\mathbf{k} = 0, \quad \mathbf{j}\cdot\mathbf{k} = 0$$

Furthermore, the vector operations discussed above may be expressed in the **ijk** notation as follows. Suppose

$$u = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \qquad \text{and} \qquad v = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

Then

$$u + v = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k} \qquad \text{and} \qquad cu = ca_1\mathbf{i} + ca_2\mathbf{j} + ca_3\mathbf{k}$$

where c is a scalar. Also,

$$u \cdot v = a_1b_1 + a_2b_2 + a_3b_3 \qquad \text{and} \qquad \|u\| = \sqrt{u \cdot u} = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

EXAMPLE 1.8 Suppose $u = 3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$ and $v = 4\mathbf{i} - 8\mathbf{j} + 7\mathbf{k}$.

(a) To find $u + v$, add corresponding components, obtaining $u + v = 7\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$

(b) To find $3u - 2v$, first multiply by the scalars and then add:

$$3u - 2v = (9\mathbf{i} + 15\mathbf{j} - 6\mathbf{k}) + (-8\mathbf{i} + 16\mathbf{j} - 14\mathbf{k}) = \mathbf{i} + 31\mathbf{j} - 20\mathbf{k}$$

(c) To find $u \cdot v$, multiply corresponding components and then add:

$$u \cdot v = 12 - 40 - 14 = -42$$

(d) To find $\|u\|$, take the square root of the sum of the squares of the components:

$$\|u\| = \sqrt{9 + 25 + 4} = \sqrt{38}$$

Cross Product

There is a special operation for vectors u and v in $\mathbf{R}^3$ that is not defined in $\mathbf{R}^n$ for $n \neq 3$. This operation is called the *cross product* and is denoted by $u \times v$. One way to easily remember the formula for $u \times v$ is to use the determinant (of order two) and its negative, which are denoted and defined as follows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad \text{and} \qquad -\begin{vmatrix} a & b \\ c & d \end{vmatrix} = bc - ad$$

Here a and d are called the *diagonal* elements and b and c are the *nondiagonal* elements. Thus, the determinant is the product ad of the diagonal elements minus the product bc of the nondiagonal elements, but vice versa for the negative of the determinant.

Now suppose $u = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $v = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$. Then

$$\begin{aligned} u \times v &= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\mathbf{k} \end{aligned}$$

That is, the three components of $u \times v$ are obtained from the array

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

(which contain the components of u above the component of v) as follows:

(1) Cover the first column and take the determinant.
(2) Cover the second column and take the negative of the determinant.
(3) Cover the third column and take the determinant.

Note that $u \times v$ is a vector; hence, $u \times v$ is also called the *vector product* or *outer product* of u and v.

EXAMPLE 1.9 Find $u \times v$ where: (a) $u = 4\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$, $v = 2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$, (b) $u = [2, -1, 5]$, $v = [3, 7, 6]$.

(a) Use $\begin{bmatrix} 4 & 3 & 6 \\ 2 & 5 & -3 \end{bmatrix}$ to get $u \times v = (-9 - 30)\mathbf{i} + (12 + 12)\mathbf{j} + (20 - 6)\mathbf{k} = -39\mathbf{i} + 24\mathbf{j} + 14\mathbf{k}$

(b) Use $\begin{bmatrix} 2 & -1 & 5 \\ 3 & 7 & 6 \end{bmatrix}$ to get $u \times v = [-6 - 35, 15 - 12, 14 + 3] = [-41, 3, 17]$

Remark: The cross products of the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are as follows:

$$\begin{aligned} &\mathbf{i} \times \mathbf{j} = \mathbf{k}, && \mathbf{j} \times \mathbf{k} = \mathbf{i}, && \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ &\mathbf{j} \times \mathbf{i} = -\mathbf{k}, && \mathbf{k} \times \mathbf{j} = -\mathbf{i}, && \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{aligned}$$

Thus, if we view the triple $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ as a cyclic permutation, where $\mathbf{i}$ follows $\mathbf{k}$ and hence $\mathbf{k}$ precedes $\mathbf{i}$, then the product of two of them in the given direction is the third one, but the product of two of them in the opposite direction is the negative of the third one.

Two important properties of the cross product are contained in the following theorem.

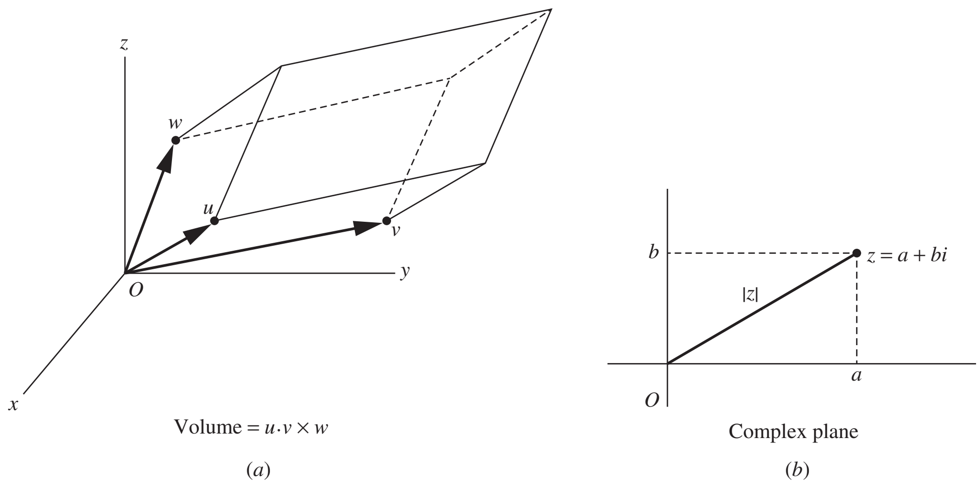


Figure 1-4

THEOREM 1.5: Let u, v, w be vectors in $\mathbf{R}^3$.

(a) The vector $u \times v$ is orthogonal to both u and v.

(b) The absolute value of the "triple product"

$$u \cdot v \times w$$

represents the volume of the parallelepiped formed by the vectors u, v, w. [See Fig. 1-4(a).]

We note that the vectors $u, v, u \times v$ form a right-handed system, and that the following formula gives the magnitude of $u \times v$:

$$\|u \times v\| = \|u\|\|v\| \sin \theta$$

where θ is the angle between u and v.

1.7 Complex Numbers

The set of complex numbers is denoted by $\mathbf{C}$. Formally, a complex number is an ordered pair (a, b) of real numbers where equality, addition, and multiplication are defined as follows:

$$(a, b) = (c, d) \quad \text{if and only if } a = c \text{ and } b = d$$
$$(a, b) + (c, d) = (a + c, \ b + d)$$
$$(a, b) \cdot (c, d) = (ac - bd, \ ad + bc)$$

We identify the real number a with the complex number $(a, 0)$; that is,

$$a \leftrightarrow (a, 0)$$

This is possible because the operations of addition and multiplication of real numbers are preserved under the correspondence; that is,

$$(a, 0) + (b, 0) = (a + b, \ 0) \qquad \text{and} \qquad (a, 0) \cdot (b, 0) = (ab, 0)$$

Thus we view $\mathbf{R}$ as a subset of $\mathbf{C}$, and replace $(a, 0)$ by a whenever convenient and possible.

We note that the set $\mathbf{C}$ of complex numbers with the above operations of addition and multiplication is a *field* of numbers, like the set $\mathbf{R}$ of real numbers and the set $\mathbf{Q}$ of *rational numbers*.

The complex number $(0, 1)$ is denoted by i. It has the important property that

$$i^2 = ii = (0,1)(0,1) = (-1,0) = -1 \qquad \text{or} \qquad i = \sqrt{-1}$$

Accordingly, any complex number $z = (a, b)$ can be written in the form

$$z = (a,b) = (a,0) + (0,b) = (a,0) + (b,0) \cdot (0,1) = a + bi$$

The above notation $z = a + bi$, where $a \equiv \text{Re}\, z$ and $b \equiv \text{Im}\, z$ are called, respectively, the *real* and *imaginary parts* of z, is more convenient than (a, b). In fact, the sum and product of complex numbers $z = a + bi$ and $w = c + di$ can be derived by simply using the commutative and distributive laws and $i^2 = -1$:

$$z + w = (a + bi) + (c + di) = a + c + bi + di = (a + b) + (c + d)i$$
$$zw = (a + bi)(c + di) = ac + bci + adi + bdi^2 = (ac - bd) + (bc + ad)i$$

We also define the *negative* of z and subtraction in $\mathbf{C}$ by

$$-z = -1z \qquad \text{and} \qquad w - z = w + (-z)$$

Warning: The letter i representing $\sqrt{-1}$ has no relationship whatsoever to the vector $\mathbf{i} = [1, 0, 0]$ in Section 1.6.

Complex Conjugate, Absolute Value

Consider a complex number $z = a + bi$. The *conjugate* of z is denoted and defined by

$$\bar{z} = \overline{a + bi} = a - bi$$

Then $z\bar{z} = (a + bi)(a - bi) = a^2 - b^2 i^2 = a^2 + b^2$. Note that z is real if and only if $\bar{z} = z$.

The *absolute value* of z, denoted by $|z|$, is defined to be the nonnegative square root of $z\bar{z}$. Namely,

$$|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$$

Note that $|z|$ is equal to the norm of the vector (a, b) in $\mathbf{R}^2$.

Suppose $z \neq 0$. Then the inverse z^{-1} of z and division in $\mathbf{C}$ of w by z are given, respectively, by

$$z^{-1} = \frac{\bar{z}}{z\bar{z}} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i \qquad \text{and} \qquad \frac{w}{z} - \frac{w\bar{z}}{z\bar{z}} = wz^{-1}$$

EXAMPLE 1.10 Suppose $z = 2 + 3i$ and $w = 5 - 2i$. Then

$$z + w = (2 + 3i) + (5 - 2i) = 2 + 5 + 3i - 2i = 7 + i$$
$$zw = (2 + 3i)(5 - 2i) = 10 + 15i - 4i - 6i^2 = 16 + 11i$$
$$\bar{z} = \overline{2 + 3i} = 2 - 3i \qquad \text{and} \qquad \bar{w} = \overline{5 - 2i} = 5 + 2i$$
$$\frac{w}{z} = \frac{5 - 2i}{2 + 3i} = \frac{(5 - 2i)(2 - 3i)}{(2 + 3i)(2 - 3i)} = \frac{4 - 19i}{13} = \frac{4}{13} - \frac{19}{13} i$$
$$|z| = \sqrt{4 + 9} = \sqrt{13} \qquad \text{and} \qquad |w| = \sqrt{25 + 4} = \sqrt{29}$$

Complex Plane

Recall that the real numbers $\mathbf{R}$ can be represented by points on a line. Analogously, the complex numbers $\mathbf{C}$ can be represented by points in the plane. Specifically, we let the point (a, b) in the plane represent the complex number $a + bi$ as shown in Fig. 1-4(b). In such a case, $|z|$ is the distance from the origin O to the point z. The plane with this representation is called the *complex plane*, just like the line representing $\mathbf{R}$ is called the *real line*.

1.8 Vectors in C^n

The set of all n-tuples of complex numbers, denoted by $\mathbf{C}^n$, is called *complex* n-*space*. Just as in the real case, the elements of $\mathbf{C}^n$ are called *points* or *vectors*, the elements of $\mathbf{C}$ are called *scalars*, and vector addition in $\mathbf{C}^n$ and scalar multiplication on $\mathbf{C}^n$ are given by

$$[z_1, z_2, \ldots, z_n] + [w_1, w_2, \ldots, w_n] = [z_1 + w_1,\ z_2 + w_2,\ \ldots,\ z_n + w_n]$$
$$z[z_1, z_2, \ldots, z_n] = [zz_1, zz_2, \ldots, zz_n]$$

where the z_i, w_i, and z belong to $\mathbf{C}$.

EXAMPLE 1.11 Consider vectors $u = [2 + 3i,\ 4 - i,\ 3]$ and $v = [3 - 2i,\ 5i,\ 4 - 6i]$ in $\mathbf{C}^3$. Then

$$u + v = [2 + 3i,\ 4 - i,\ 3] + [3 - 2i,\ 5i,\ 4 - 6i] = [5 + i,\ 4 + 4i,\ 7 - 6i]$$
$$(5 - 2i)u = [(5 - 2i)(2 + 3i),\ (5 - 2i)(4 - i),\ (5 - 2i)(3)] = [16 + 11i,\ 18 - 13i,\ 15 - 6i]$$

Dot (Inner) Product in C^n

Consider vectors $u = [z_1, z_2, \ldots, z_n]$ and $v = [w_1, w_2, \ldots, w_n]$ in $\mathbf{C}^n$. The *dot* or *inner product* of u and v is denoted and defined by

$$u \cdot v = z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n$$

This definition reduces to the real case because $\bar{w}_i = w_i$ when w_i is real. The norm of u is defined by

$$\|u\| = \sqrt{u \cdot u} = \sqrt{z_1\bar{z}_1 + z_2\bar{z}_2 + \cdots + z_n\bar{z}_n} = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |v_n|^2}$$

We emphasize that $u \cdot u$ and so $\|u\|$ are real and positive when $u \neq 0$ and 0 when $u = 0$.

EXAMPLE 1.12 Consider vectors $u = [2 + 3i,\ 4 - i,\ 3 + 5i]$ and $v = [3 - 4i,\ 5i,\ 4 - 2i]$ in $\mathbf{C}^3$. Then

$$\begin{aligned}
u \cdot v &= (2 + 3i)(\overline{3 - 4i}) + (4 - i)(\overline{5i}) + (3 + 5i)(\overline{4 - 2i}) \\
&= (2 + 3i)(3 + 4i) + (4 - i)(-5i) + (3 + 5i)(4 + 2i) \\
&= (-6 + 13i) + (-5 - 20i) + (2 + 26i) = -9 + 19i \\
u \cdot u &= |2 + 3i|^2 + |4 - i|^2 + |3 + 5i|^2 = 4 + 9 + 16 + 1 + 9 + 25 = 64 \\
\|u\| &= \sqrt{64} = 8
\end{aligned}$$

The space $\mathbf{C}^n$ with the above operations of vector addition, scalar multiplication, and dot product, is called *complex Euclidean* n-*space*. Theorem 1.2 for $\mathbf{R}^n$ also holds for $\mathbf{C}^n$ if we replace $u \cdot v = v \cdot u$ by

$$u \cdot v = \overline{u \cdot v}$$

On the other hand, the Schwarz inequality (Theorem 1.3) and Minkowski's inequality (Theorem 1.4) are true for $\mathbf{C}^n$ with no changes.

SOLVED PROBLEMS

Vectors in R^n

1.1. Determine which of the following vectors are equal:

$$u_1 = (1, 2, 3), \qquad u_2 = (2, 3, 1), \qquad u_3 = (1, 3, 2), \qquad u_4 = (2, 3, 1)$$

Vectors are equal only when corresponding entries are equal; hence, only $u_2 = u_4$.

1.2. Let $u = (2, -7, 1)$, $v = (-3, 0, 4)$, $w = (0, 5, -8)$. Find:

(a) $3u - 4v$,

(b) $2u + 3v - 5w$.

First perform the scalar multiplication and then the vector addition.

(a) $3u - 4v = 3(2, -7, 1) - 4(-3, 0, 4) = (6, -21, 3) + (12, 0, -16) = (18, -21, -13)$

(b) $2u + 3v - 5w = (4, -14, 2) + (-9, 0, 12) + (0, -25, 40) = (-5, -39, 54)$

1.3. Let $u = \begin{bmatrix} 5 \\ 3 \\ -4 \end{bmatrix}, v = \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}, w = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$. Find:

(a) $5u - 2v$,

(b) $-2u + 4v - 3w$.

First perform the scalar multiplication and then the vector addition:

(a) $5u - 2v = 5\begin{bmatrix} 5 \\ 3 \\ -4 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 25 \\ 15 \\ -20 \end{bmatrix} + \begin{bmatrix} 2 \\ -10 \\ -4 \end{bmatrix} = \begin{bmatrix} 27 \\ 5 \\ -24 \end{bmatrix}$

(b) $-2u + 4v - 3w = \begin{bmatrix} -10 \\ -6 \\ 8 \end{bmatrix} + \begin{bmatrix} -4 \\ 20 \\ 8 \end{bmatrix} + \begin{bmatrix} -9 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -23 \\ 17 \\ 22 \end{bmatrix}$

1.4. Find x and y, where: (a) $(x, 3) = (2,\ x + y)$, (b) $(4, y) = x(2, 3)$.

(a) Because the vectors are equal, set the corresponding entries equal to each other, yielding

$$x = 2, \qquad 3 = x + y$$

Solve the linear equations, obtaining $x = 2$, $y = 1$.

(b) First multiply by the scalar x to obtain $(4, y) = (2x, 3x)$. Then set corresponding entries equal to each other to obtain

$$4 = 2x, \qquad y = 3x$$

Solve the equations to yield $x = 2$, $y = 6$.

1.5. Write the vector $v = (1, -2, 5)$ as a linear combination of the vectors $u_1 = (1, 1, 1)$, $u_2 = (1, 2, 3)$, $u_3 = (2, -1, 1)$.

We want to express v in the form $v = xu_1 + yu_2 + zu_3$ with x, y, z as yet unknown. First we have

$$\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + z\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x + y + 2z \\ x + 2y - z \\ x + 3y + z \end{bmatrix}$$

(It is more convenient to write vectors as columns than as rows when forming linear combinations.) Set corresponding entries equal to each other to obtain

$$\begin{aligned} x + y + 2z &= 1 \\ x + 2y - z &= -2 \\ x + 3y + z &= 5 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + y + 2z &= 1 \\ y - 3z &= -3 \\ 2y - z &= 4 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + y + 2z &= 1 \\ y - 3z &= -3 \\ 5z &= 10 \end{aligned}$$

This unique solution of the triangular system is $x = -6$, $y = 3$, $z = 2$. Thus, $v = -6u_1 + 3u_2 + 2u_3$.

1.6. Write $v = (2, -5, 3)$ as a linear combination of

$$u_1 = (1, -3, 2), u_2 = (2, -4, -1), u_3 = (1, -5, 7).$$

Find the equivalent system of linear equations and then solve. First,

$$\begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix} = x\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + y\begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix} + z\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} x + 2y + z \\ -3x - 4y - 5z \\ 2x - y + 7z \end{bmatrix}$$

Set the corresponding entries equal to each other to obtain

$$\begin{aligned} x + 2y + z &= 2 \\ -3x - 4y - 5z &= -5 \\ 2x - y + 7z &= 3 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + 2y + z &= 2 \\ 2y - 2z &= 1 \\ -5y + 5z &= -1 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + 2y + z &= 2 \\ 2y - 2z &= 1 \\ 0 &= 3 \end{aligned}$$

The third equation, $0x + 0y + 0z = 3$, indicates that the system has no solution. Thus, v cannot be written as a linear combination of the vectors u_1, u_2, u_3.

Dot (Inner) Product, Orthogonality, Norm in R^n

1.7. Find $u \cdot v$ where:

(a) $u = (2, -5, 6)$ and $v = (8, 2, -3)$,

(b) $u = (4, 2, -3, 5, -1)$ and $v = (2, 6, -1, -4, 8)$.

Multiply the corresponding components and add:

(a) $u \cdot v = 2(8) - 5(2) + 6(-3) = 16 - 10 - 18 = -12$

(b) $u \cdot v = 8 + 12 + 3 - 20 - 8 = -5$

1.8. Let $u = (5, 4, 1)$, $v = (3, -4, 1)$, $w = (1, -2, 3)$. Which pair of vectors, if any, are perpendicular (orthogonal)?

Find the dot product of each pair of vectors:

$$u \cdot v = 15 - 16 + 1 = 0, \qquad v \cdot w = 3 + 8 + 3 = 14, \qquad u \cdot w = 5 - 8 + 3 = 0$$

Thus, u and v are orthogonal, u and w are orthogonal, but v and w are not.

1.9. Find k so that u and v are orthogonal, where:

(a) $u = (1, k, -3)$ and $v = (2, -5, 4)$,

(b) $u = (2, 3k, -4, 1, 5)$ and $v = (6, -1, 3, 7, 2k)$.

Compute $u \cdot v$, set $u \cdot v$ equal to 0, and then solve for k:

(a) $u \cdot v = 1(2) + k(-5) - 3(4) = -5k - 10$. Then $-5k - 10 = 0$, or $k = -2$.

(b) $u \cdot v = 12 - 3k - 12 + 7 + 10k = 7k + 7$. Then $7k + 7 = 0$, or $k = -1$.

1.10. Find $\|u\|$, where: (a) $u = (3, -12, -4)$, (b) $u = (2, -3, 8, -7)$.

First find $\|u\|^2 = u \cdot u$ by squaring the entries and adding. Then $\|u\| = \sqrt{\|u\|^2}$.

(a) $\|u\|^2 = (3)^2 + (-12)^2 + (-4)^2 = 9 + 144 + 16 = 169$. Then $\|u\| = \sqrt{169} = 13$.

(b) $\|u\|^2 = 4 + 9 + 64 + 49 = 126$. Then $\|u\| = \sqrt{126}$.

1.11. Recall that *normalizing* a nonzero vector v means finding the unique unit vector $\hat{v}$ in the same direction as v, where

$$\hat{v} = \frac{1}{\|v\|} v$$

Normalize: (a) $u = (3, -4)$, (b) $v = (4, -2, -3, 8)$, (c) $w = (\frac{1}{2}, \frac{2}{3}, -\frac{1}{4})$.

(a) First find $\|u\| = \sqrt{9+16} = \sqrt{25} = 5$. Then divide each entry of u by 5, obtaining $\hat{u} = (\frac{3}{5}, -\frac{4}{5})$.

(b) Here $\|v\| = \sqrt{16+4+9+64} = \sqrt{93}$. Then

$$\hat{v} = \left(\frac{4}{\sqrt{93}}, \frac{-2}{\sqrt{93}}, \frac{-3}{\sqrt{93}}, \frac{8}{\sqrt{93}}\right)$$

(c) Note that w and any positive multiple of w will have the same normalized form. Hence, first multiply w by 12 to "clear fractions"—that is, first find $w' = 12w = (6, 8, -3)$. Then

$$\|w'\| = \sqrt{36+64+9} = \sqrt{109} \quad \text{and} \quad \hat{w} = \widehat{w'} = \left(\frac{6}{\sqrt{109}}, \frac{8}{\sqrt{109}}, \frac{-3}{\sqrt{109}}\right)$$

1.12. Let $u = (1, -3, 4)$ and $v = (3, 4, 7)$. Find:

(a) $\cos\theta$, where θ is the angle between u and v;

(b) $\text{proj}(u, v)$, the projection of u onto v;

(c) $d(u, v)$, the distance between u and v.

First find $u \cdot v = 3 - 12 + 28 = 19$, $\|u\|^2 = 1 + 9 + 16 = 26$, $\|v\|^2 = 9 + 16 + 49 = 74$. Then

(a) $\cos\theta = \dfrac{u \cdot v}{\|u\|\|v\|} = \dfrac{19}{\sqrt{26}\sqrt{74}}$,

(b) $\text{proj}(u, v) = \dfrac{u \cdot v}{\|v\|^2} v = \dfrac{19}{74}(3, 4, 7) = \left(\dfrac{57}{74}, \dfrac{76}{74}, \dfrac{133}{74}\right) = \left(\dfrac{57}{74}, \dfrac{38}{37}, \dfrac{133}{74}\right)$,

(c) $d(u, v) = \|u - v\| = \|(-2, -7 - 3)\| = \sqrt{4 + 49 + 9} = \sqrt{62}$.

1.13. Prove Theorem 1.2: For any u, v, w in $\mathbf{R}^n$ and k in $\mathbf{R}$:

(i) $(u + v) \cdot w = u \cdot w + v \cdot w$, (ii) $(ku) \cdot v = k(u \cdot v)$, (iii) $u \cdot v = v \cdot u$,
(iv) $u \cdot u \geq 0$, and $u \cdot u = 0$ iff $u = 0$.

Let $u = (u_1, u_2, \ldots, u_n)$, $v = (v_1, v_2, \ldots, v_n)$, $w = (w_1, w_2, \ldots, w_n)$.

(i) Because $u + v = (u_1 + v_1,\ u_2 + v_2,\ \ldots,\ u_n + v_n)$,

$$\begin{aligned}(u+v)\cdot w &= (u_1+v_1)w_1 + (u_2+v_2)w_2 + \cdots + (u_n+v_n)w_n \\ &= u_1w_1 + v_1w_1 + u_2w_2 + \cdots + u_nw_n + v_nw_n \\ &= (u_1w_1 + u_2w_2 + \cdots + u_nw_n) + (v_1w_1 + v_2w_2 + \cdots + v_nw_n) \\ &= u\cdot w + v\cdot w\end{aligned}$$

(ii) Because $ku = (ku_1, ku_2, \ldots, ku_n)$,

$$(ku)\cdot v = ku_1v_1 + ku_2v_2 + \cdots + ku_nv_n = k(u_1v_1 + u_2v_2 + \cdots + u_nv_n) = k(u\cdot v)$$

(iii) $u \cdot v = u_1v_1 + u_2v_2 + \cdots + u_nv_n = v_1u_1 + v_2u_2 + \cdots + v_nu_n = v \cdot u$

(iv) Because u_i^2 is nonnegative for each i, and because the sum of nonnegative real numbers is nonnegative,

$$u\cdot u = u_1^2 + u_2^2 + \cdots + u_n^2 \geq 0$$

Furthermore, $u \cdot u = 0$ iff $u_i = 0$ for each i, that is, iff $u = 0$.

1.14. Prove Theorem 1.3 (Schwarz): $|u \cdot v| \leq \|u\|\|v\|$.

For any real number t, and using Theorem 1.2, we have

$$0 \leq (tu + v) \cdot (tu + v) = t^2(u \cdot u) + 2t(u \cdot v) + (v \cdot v) = \|u\|^2 t^2 + 2(u \cdot v)t + \|v\|^2$$

Let $a = \|u\|^2$, $b = 2(u \cdot v)$, $c = \|v\|^2$. Then, for every value of t, $at^2 + bt + c \geq 0$. This means that the quadratic polynomial cannot have two real roots. This implies that the discriminant $D = b^2 - 4ac \leq 0$ or, equivalently, $b^2 \leq 4ac$. Thus,

$$4(u \cdot v)^2 \leq 4\|u\|^2\|v\|^2$$

Dividing by 4 gives us our result.

1.15. Prove Theorem 1.4 (Minkowski): $\|u + v\| \leq \|u\| + \|v\|$.

By the Schwarz inequality and other properties of the dot product,

$$\|u + v\|^2 = (u + v) \cdot (u + v) = (u \cdot u) + 2(u \cdot v) + (v \cdot v) \leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 = (\|u\| + \|v\|)^2$$

Taking the square root of both sides yields the desired inequality.

Points, Lines, Hyperplanes in R^n

Here we distinguish between an n-tuple $P(a_1, a_2, \ldots, a_n)$ viewed as a point in $\mathbf{R}^n$ and an n-tuple $u = [c_1, c_2, \ldots, c_n]$ viewed as a vector (arrow) from the origin O to the point $C(c_1, c_2, \ldots, c_n)$.

1.16. Find the vector u identified with the directed line segment $\overrightarrow{PQ}$ for the points:

(a) $P(1, -2, 4)$ and $Q(6, 1, -5)$ in $\mathbf{R}^3$, (b) $P(2, 3, -6, 5)$ and $Q(7, 1, 4, -8)$ in $\mathbf{R}^4$.

(a) $u = \overrightarrow{PQ} = Q - P = [6 - 1,\ 1 - (-2),\ -5 - 4] = [5, 3, -9]$

(b) $u = \overrightarrow{PQ} = Q - P = [7 - 2,\ 1 - 3,\ 4 + 6,\ -8 - 5] = [5, -2, 10, -13]$

1.17. Find an equation of the hyperplane H in $\mathbf{R}^4$ that passes through $P(3, -4, 1, -2)$ and is normal to $u = [2, 5, -6, -3]$.

The coefficients of the unknowns of an equation of H are the components of the normal vector u. Thus, an equation of H is of the form $2x_1 + 5x_2 - 6x_3 - 3x_4 = k$. Substitute P into this equation to obtain $k = -26$. Thus, an equation of H is $2x_1 + 5x_2 - 6x_3 - 3x_4 = -26$.

1.18. Find an equation of the plane H in $\mathbf{R}^3$ that contains $P(1, -3, -4)$ and is parallel to the plane H' determined by the equation $3x - 6y + 5z = 2$.

The planes H and H' are parallel if and only if their normal directions are parallel or antiparallel (opposite direction). Hence, an equation of H is of the form $3x - 6y + 5z = k$. Substitute P into this equation to obtain $k = 1$. Then an equation of H is $3x - 6y + 5z = 1$.

1.19. Find a parametric representation of the line L in $\mathbf{R}^4$ passing through $P(4, -2, 3, 1)$ in the direction of $u = [2, 5, -7, 8]$.

Here L consists of the points $X(x_i)$ that satisfy

$$X = P + tu \quad \text{or} \quad x_i = a_i t + b_i \quad \text{or} \quad L(t) = (a_i t + b_i)$$

where the parameter t takes on all real values. Thus we obtain

$$x_1 = 4 + 2t,\ x_2 = -2 + 2t,\ x_3 = 3 - 7t,\ x_4 = 1 + 8t \quad \text{or} \quad L(t) = (4 + 2t,\ -2 + 2t,\ 3 - 7t,\ 1 + 8t)$$

1.20. Let C be the curve $F(t) = (t^2,\ 3t - 2,\ t^3,\ t^2 + 5)$ in $\mathbf{R}^4$, where $0 \le t \le 4$.

(a) Find the point P on C corresponding to $t = 2$.

(b) Find the initial point Q and terminal point Q' of C.

(c) Find the unit tangent vector $\mathbf{T}$ to the curve C when $t = 2$.

(a) Substitute $t = 2$ into $F(t)$ to get $P = f(2) = (4, 4, 8, 9)$.

(b) The parameter t ranges from $t = 0$ to $t = 4$. Hence, $Q = f(0) = (0, -2, 0, 5)$ and $Q' = F(4) = (16, 10, 64, 21)$.

(c) Take the derivative of $F(t)$—that is, of each component of $F(t)$—to obtain a vector V that is tangent to the curve:

$$V(t) = \frac{dF(t)}{dt} = [2t, 3, 3t^2, 2t]$$

Now find V when $t = 2$; that is, substitute $t = 2$ in the equation for $V(t)$ to obtain $V = V(2) = [4, 3, 12, 4]$. Then normalize V to obtain the desired unit tangent vector $\mathbf{T}$. We have

$$\|V\| = \sqrt{16 + 9 + 144 + 16} = \sqrt{185} \qquad \text{and} \qquad \mathbf{T} = \left[\frac{4}{\sqrt{185}}, \frac{3}{\sqrt{185}}, \frac{12}{\sqrt{185}}, \frac{4}{\sqrt{185}}\right]$$

Spatial Vectors (Vectors in $\mathbf{R}^3$), ijk Notation, Cross Product

1.21. Let $u = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, $v = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $w = \mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$. Find:

(a) $u + v$, (b) $2u - 3v + 4w$, (c) $u \cdot v$ and $u \cdot w$, (d) $\|u\|$ and $\|v\|$.

Treat the coefficients of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ just like the components of a vector in $\mathbf{R}^3$.

(a) Add corresponding coefficients to get $u + v = 5\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$.

(b) First perform the scalar multiplication and then the vector addition:

$$\begin{aligned} 2u - 3v + 4w &= (4\mathbf{i} - 6\mathbf{j} + 8\mathbf{k}) + (-9\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) + (4\mathbf{i} + 20\mathbf{j} + 12\mathbf{k}) \\ &= -\mathbf{i} + 11\mathbf{j} + 26\mathbf{k} \end{aligned}$$

(c) Multiply corresponding coefficients and then add:

$$u \cdot v = 6 - 3 - 8 = -5 \qquad \text{and} \qquad u \cdot w = 2 - 15 + 12 = -1$$

(d) The norm is the square root of the sum of the squares of the coefficients:

$$\|u\| = \sqrt{4 + 9 + 16} = \sqrt{29} \qquad \text{and} \qquad \|v\| = \sqrt{9 + 1 + 4} = \sqrt{14}$$

1.22. Find the (parametric) equation of the line L:

(a) through the points $P(1, 3, 2)$ and $Q(2, 5, -6)$;

(b) containing the point $P(1, -2, 4)$ and perpendicular to the plane H given by the equation $3x + 5y + 7z = 15$.

(a) First find $v = \overrightarrow{PQ} = Q - P = [1, 2, -8] = \mathbf{i} + 2\mathbf{j} - 8\mathbf{k}$. Then

$$L(t) = (t + 1,\ 2t + 3,\ -8t + 2) = (t + 1)\mathbf{i} + (2t + 3)\mathbf{j} + (-8t + 2)\mathbf{k}$$

(b) Because L is perpendicular to H, the line L is in the same direction as the normal vector $\mathbf{N} = 3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$ to H. Thus,

$$L(t) = (3t + 1,\ 5t - 2,\ 7t + 4) = (3t + 1)\mathbf{i} + (5t - 2)\mathbf{j} + (7t + 4)\mathbf{k}$$

1.23. Let S be the surface $xy^2 + 2yz = 16$ in $\mathbf{R}^3$.

(a) Find the normal vector $\mathbf{N}(x, y, z)$ to the surface S.

(b) Find the tangent plane H to S at the point $P(1, 2, 3)$.

(a) The formula for the normal vector to a surface $F(x, y, z) = 0$ is

$$\mathbf{N}(x, y, z) = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$

where F_x, F_y, F_z are the partial derivatives. Using $F(x, y, z) = xy^2 + 2yz - 16$, we obtain

$$F_x = y^2, \qquad F_y = 2xy + 2z, \qquad F_z = 2y$$

Thus, $\mathbf{N}(x, y, z) = y^2\mathbf{i} + (2xy + 2z)\mathbf{j} + 2y\mathbf{k}$.

(b) The normal to the surface S at the point P is

$$\mathbf{N}(P) = \mathbf{N}(1, 2, 3) = 4\mathbf{i} + 10\mathbf{j} + 4\mathbf{k}$$

Hence, $\mathbf{N} = 2\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$ is also normal to S at P. Thus an equation of H has the form $2x + 5y + 2z = c$. Substitute P in this equation to obtain $c = 18$. Thus the tangent plane H to S at P is $2x + 5y + 2z = 18$.

1.24. Evaluate the following determinants and negative of determinants of order two:

(a) (i) $\begin{vmatrix} 3 & 4 \\ 5 & 9 \end{vmatrix}$, (ii) $\begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix}$, (iii) $\begin{vmatrix} 4 & -5 \\ 3 & -2 \end{vmatrix}$

(b) (i) $-\begin{vmatrix} 3 & 6 \\ 4 & 2 \end{vmatrix}$, (ii) $-\begin{vmatrix} 7 & -5 \\ 3 & 2 \end{vmatrix}$, (iii) $-\begin{vmatrix} 4 & -1 \\ 8 & -3 \end{vmatrix}$

Use $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ and $-\begin{vmatrix} a & b \\ c & d \end{vmatrix} = bc - ad$. Thus,

(a) (i) $27 - 20 = 7$, (ii) $6 + 4 = 10$, (iii) $-8 + 15 = 7$.

(b) (i) $24 - 6 = 18$, (ii) $-15 - 14 = -29$, (iii) $-8 + 12 = 4$.

1.25. Let $u = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, $v = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $w = \mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$.
Find: (a) $u \times v$, (*b*) $u \times w$

(a) Use $\begin{bmatrix} 2 & -3 & 4 \\ 3 & 1 & -2 \end{bmatrix}$ to get $u \times v = (6 - 4)\mathbf{i} + (12 + 4)\mathbf{j} + (2 + 9)\mathbf{k} = 2\mathbf{i} + 16\mathbf{j} + 11\mathbf{k}$.

(b) Use $\begin{bmatrix} 2 & -3 & 4 \\ 1 & 5 & 3 \end{bmatrix}$ to get $u \times w = (-9 - 20)\mathbf{i} + (4 - 6)\mathbf{j} + (10 + 3)\mathbf{k} = -29\mathbf{i} - 2\mathbf{j} + 13\mathbf{k}$.

1.26. Find $u \times v$, where: (a) $u = (1, 2, 3)$, $v = (4, 5, 6)$; (b) $u = (-4, 7, 3)$, $v = (6, -5, 2)$.

(a) Use $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ to get $u \times v = [12 - 15,\ 12 - 6,\ 5 - 8] = [-3, 6, -3]$.

(b) Use $\begin{bmatrix} -4 & 7 & 3 \\ 6 & -5 & 2 \end{bmatrix}$ to get $u \times v = [14 + 15,\ 18 + 8,\ 20 - 42] = [29, 26, -22]$.

1.27. Find a unit vector u orthogonal to $v = [1, 3, 4]$ and $w = [2, -6, -5]$.

First find $v \times w$, which is orthogonal to v and w.

The array $\begin{bmatrix} 1 & 3 & 4 \\ 2 & -6 & -5 \end{bmatrix}$ gives $v \times w = [-15 + 24,\ 8 + 5,\ -6 - 61] = [9, 13, -12]$.

Normalize $v \times w$ to get $u = [9/\sqrt{394},\ 13/\sqrt{394},\ -12/\sqrt{394}]$.

1.28. Let $u = (a_1, a_2, a_3)$ and $v = (b_1, b_2, b_3)$ so $u \times v = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$.
Prove:

(a) $u \times v$ is orthogonal to u and v [Theorem 1.5(a)].

(b) $\|u \times v\|^2 = (u \cdot u)(v \cdot v) - (u \cdot v)^2$ (Lagrange's identity).

(a) We have

$$u \cdot (u \times v) = a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1)$$
$$= a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_1a_2b_3 + a_1a_3b_2 - a_2a_3b_1 = 0$$

Thus, $u \times v$ is orthogonal to u. Similarly, $u \times v$ is orthogonal to v.

(b) We have

$$\|u \times v\|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \quad (1)$$
$$(u \cdot u)(v \cdot v) - (u \cdot v)^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \quad (2)$$

Expansion of the right-hand sides of (1) and (2) establishes the identity.

Complex Numbers, Vectors in C^n

1.29. Suppose $z = 5 + 3i$ and $w = 2 - 4i$. Find: (a) $z + w$, (b) $z - w$, (c) zw.

Use the ordinary rules of algebra together with $i^2 = -1$ to obtain a result in the standard form $a + bi$.

(a) $z + w = (5 + 3i) + (2 - 4i) = 7 - i$
(b) $z - w = (5 + 3i) - (2 - 4i) = 5 + 3i - 2 + 4i = 3 + 7i$
(c) $zw = (5 + 3i)(2 - 4i) = 10 - 14i - 12i^2 = 10 - 14i + 12 = 22 - 14i$

1.30. Simplify: (a) $(5 + 3i)(2 - 7i)$, (b) $(4 - 3i)^2$, (c) $(1 + 2i)^3$.

(a) $(5 + 3i)(2 - 7i) = 10 + 6i - 35i - 21i^2 = 31 - 29i$
(b) $(4 - 3i)^2 = 16 - 24i + 9i^2 = 7 - 24i$
(c) $(1 + 2i)^3 = 1 + 6i + 12i^2 + 8i^3 = 1 + 6i - 12 - 8i = -11 - 2i$

1.31. Simplify: (a) i^0, i^3, i^4, (b) i^5, i^6, i^7, i^8, (c) i^{39}, i^{174}, i^{252}, i^{317}.

(a) $i^0 = 1$, $i^3 = i^2(i) = (-1)(i) = -i$, $i^4 = (i^2)(i^2) = (-1)(-1) = 1$
(b) $i^5 = (i^4)(i) = (1)(i) = i$, $i^6 = (i^4)(i^2) = (1)(i^2) = i^2 = -1$, $i^7 = i^3 = -i$, $i^8 = i^4 = 1$
(c) Using $i^4 = 1$ and $i^n = i^{4q+r} = (i^4)^q i^r = 1^q i^r = i^r$, divide the exponent n by 4 to obtain the remainder r:

$$i^{39} = i^{4(9)+3} = (i^4)^9 i^3 = 1^9 i^3 = i^3 = -i, \qquad i^{174} = i^2 = -1, \qquad i^{252} = i^0 = 1, \qquad i^{317} = i^1 = i$$

1.32. Find the complex conjugate of each of the following:
(a) $6 + 4i$, $7 - 5i$, $4 + i$, $-3 - i$, (b) 6, -3, $4i$, $-9i$.

(a) $\overline{6 + 4i} = 6 - 4i$, $\overline{7 - 5i} = 7 + 5i$, $\overline{4 + i} = 4 - i$, $\overline{-3 - i} = -3 + i$
(b) $\bar{6} = 6$, $\overline{-3} = -3$, $\overline{4i} = -4i$, $\overline{-9i} = 9i$

(Note that the conjugate of a real number is the original number, but the conjugate of a pure imaginary number is the negative of the original number.)

1.33. Find $z\bar{z}$ and $|z|$ when $z = 3 + 4i$.

For $z = a + bi$, use $z\bar{z} = a^2 + b^2$ and $z = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$.

$$z\bar{z} = 9 + 16 = 25, \qquad |z| = \sqrt{25} = 5$$

1.34. Simpify $\dfrac{2 - 7i}{5 + 3i}$.

To simplify a fraction z/w of complex numbers, multiply both numerator and denominator by $\bar{w}$, the conjugate of the denominator:

$$\frac{2 - 7i}{5 + 3i} = \frac{(2 - 7i)(5 - 3i)}{(5 + 3i)(5 - 3i)} = \frac{-11 - 41i}{34} = -\frac{11}{34} - \frac{41}{34}i$$

1.35. Prove: For any complex numbers $z, w \in \mathbf{C}$, (i) $\overline{z+w} = \bar{z} + \bar{w}$, (ii) $\overline{zw} = \bar{z}\bar{w}$, (iii) $\bar{\bar{z}} = z$.

Suppose $z = a + bi$ and $w = c + di$ where $a, b, c, d \in \mathbf{R}$.

(i) $$\begin{aligned}\overline{z+w} &= \overline{(a+bi)+(c+di)} = \overline{(a+c)+(b+d)i} \\ &= (a+c)-(b+d)i = a+c-bi-di \\ &= (a-bi)+(c-di) = \bar{z}+\bar{w}\end{aligned}$$

(ii) $$\begin{aligned}\overline{zw} &= \overline{(a+bi)(c+di)} = \overline{(ac-bd)+(ad+bc)i} \\ &= (ac-bd)-(ad+bc)i = (a-bi)(c-di) = \bar{z}\bar{w}\end{aligned}$$

(iii) $\bar{\bar{z}} = \overline{\overline{a+bi}} = \overline{a-bi} = a-(-b)i = a+bi = z$

1.36. Prove: For any complex numbers $z, w \in \mathbf{C}$, $|zw| = |z||w|$.

By (ii) of Problem 1.35,

$$|zw|^2 = (zw)(\overline{zw}) = (zw)(\bar{z}\bar{w}) = (z\bar{z})(w\bar{w}) = |z|^2|w|^2$$

The square root of both sides gives us the desired result.

1.37. Prove: For any complex numbers $z, w \in \mathbf{C}$, $|z+w| \le |z| + |w|$.

Suppose $z = a + bi$ and $w = c + di$ where $a, b, c, d \in \mathbf{R}$. Consider the vectors $u = (a, b)$ and $v = (c, d)$ in $\mathbf{R}^2$. Note that

$$|z| = \sqrt{a^2+b^2} = \|u\|, \qquad |w| = \sqrt{c^2+d^2} = \|v\|$$

and

$$|z+w| = |(a+c)+(b+d)i| = \sqrt{(a+c)^2+(b+d)^2} = \|(a+c, b+d)\| = \|u+v\|$$

By Minkowski's inequality (Problem 1.15), $\|u+v\| \le \|u\| + \|v\|$, and so

$$|z+w| = \|u+v\| \le \|u\| + \|v\| = |z| + |w|$$

1.38. Find the dot products $u \cdot v$ and $v \cdot u$ where: (a) $u = (1-2i,\ 3+i)$, $v = (4+2i,\ 5-6i)$, (b) $u = (3-2i,\ 4i,\ 1+6i)$, $v = (5+i,\ 2-3i,\ 7+2i)$.

Recall that conjugates of the second vector appear in the dot product

$$(z_1, \ldots, z_n) \cdot (w_1, \ldots, w_n) = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$$

(a) $$\begin{aligned}u \cdot v &= (1-2i)\overline{(4+2i)} + (3+i)\overline{(5-6i)} \\ &= (1-2i)(4-2i) + (3+i)(5+6i) = -10i+9+23i = 9+13i \\ v \cdot u &= (4+2i)\overline{(1-2i)} + (5-6i)\overline{(3+i)} \\ &= (4+2i)(1+2i) + (5-6i)(3-i) = 10i+9-23i = 9-13i\end{aligned}$$

(b) $$\begin{aligned}u \cdot v &= (3-2i)\overline{(5+i)} + (4i)\overline{(2-3i)} + (1+6i)\overline{(7+2i)} \\ &= (3-2i)(5-i) + (4i)(2+3i) + (1+6i)(7-2i) = 20+35i \\ v \cdot u &= (5+i)\overline{(3-2i)} + (2-3i)\overline{(4i)} + (7+2i)\overline{(1+6i)} \\ &= (5+i)(3+2i) + (2-3i)(-4i) + (7+2i)(1-6i) = 20-35i\end{aligned}$$

In both cases, $v \cdot u = \overline{u \cdot v}$. This holds true in general, as seen in Problem 1.40.

1.39. Let $u = (7-2i,\ 2+5i)$ and $v = (1+i,\ -3-6i)$. Find:
(a) $u+v$, (b) $2iu$, (c) $(3-i)v$, (d) $u \cdot v$, (e) $\|u\|$ and $\|v\|$.

(a) $u + v = (7-2i+1+i,\ 2+5i-3-6i) = (8-i,\ -1-i)$

(b) $2iu = (14i-4i^2,\ 4i+10i^2) = (4+14i,\ -10+4i)$

(c) $(3-i)v = (3+3i-i-i^2,\ -9-18i+3i+6i^2) = (4+2i,\ -15-15i)$

(d) $u \cdot v = (7-2i)(\overline{1+i}) + (2+5i)(\overline{-3-6i})$
$= (7-2i)(1-i) + (2+5i)(-3+6i) \;=\; 5-9i-36-3i \;=\; -31-12i$

(e) $\|u\| = \sqrt{7^2 + (-2)^2 + 2^2 + 5^2} = \sqrt{82}$ and $\|v\| = \sqrt{1^2 + 1^2 + (-3)^2 + (-6)^2} = \sqrt{47}$

1.40. Prove: For any vectors $u, v \in \mathbf{C}^n$ and any scalar $z \in \mathbf{C}$, (i) $u \cdot v = \overline{v \cdot u}$, (ii) $(zu) \cdot v = z(u \cdot v)$, (iii) $u \cdot (zv) = \bar{z}(u \cdot v)$.

Suppose $u = (z_1, z_2, \ldots, z_n)$ and $v = (w_1, w_2, \ldots, w_n)$.

(i) Using the properties of the conjugate,

$$\overline{v \cdot u} = \overline{w_1\bar{z}_1 + w_2\bar{z}_2 + \cdots + w_n\bar{z}_n} = \overline{w_1\bar{z}_1} + \overline{w_2\bar{z}_2} + \cdots + \overline{w_n\bar{z}_n}$$
$$= \bar{w}_1 z_1 + \bar{w}_2 z_2 + \cdots + \bar{w}_n z_n = z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n = u \cdot v$$

(ii) Because $zu = (zz_1, zz_2, \ldots, zz_n)$,

$$(zu) \cdot v = zz_1\bar{w}_1 + zz_2\bar{w}_2 + \cdots + zz_n\bar{w}_n = z(z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n) = z(u \cdot v)$$

(Compare with Theorem 1.2 on vectors in $\mathbf{R}^n$.)

(iii) Using (i) and (ii),

$$u \cdot (zv) = \overline{(zv) \cdot u} = \overline{z(\overline{v \cdot u})} = \bar{z}(\overline{v \cdot u}) = \bar{z}(u \cdot v)$$

SUPPLEMENTARY PROBLEMS

Vectors in R^n

1.41. Let $u = (1, -2, 4)$, $v = (3, 5, 1)$, $w = (2, 1, -3)$. Find:

(a) $3u - 2v$; (b) $5u + 3v - 4w$; (c) $u \cdot v$, $u \cdot w$, $v \cdot w$; (d) $\|u\|$, $\|v\|$, $\|w\|$;
(e) $\cos\theta$, where θ is the angle between u and v; (f) $d(u, v)$; (g) $\text{proj}(u, v)$.

1.42. Repeat Problem 1.41 for vectors $u = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$, $w = \begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix}$.

1.43. Let $u = (2, -5, 4, 6, -3)$ and $v = (5, -2, 1, -7, -4)$. Find:

(a) $4u - 3v$; (b) $5u + 2v$; (c) $u \cdot v$; (d) $\|u\|$ and $\|v\|$; (e) $\text{proj}(u, v)$; (f) $d(u, v)$.

1.44. Normalize each vector:

(a) $u = (5, -7)$; (b) $v = (1, 2, -2, 4)$; (c) $w = \left(\frac{1}{2}, -\frac{1}{3}, \frac{3}{4}\right)$.

1.45. Let $u = (1, 2, -2)$, $v = (3, -12, 4)$, and $k = -3$.

(a) Find $\|u\|$, $\|v\|$, $\|u + v\|$, $\|ku\|$.
(b) Verify that $\|ku\| = |k|\|u\|$ and $\|u + v\| \le \|u\| + \|v\|$.

1.46. Find x and y where:

(a) $(x,\; y+1) = (y-2,\; 6)$; (b) $x(2, y) = y(1, -2)$.

1.47. Find x, y, z where $(x,\; y+1,\; y+z) \;=\; (2x+y,\; 4,\; 3z)$.

1.48. Write $v = (2, 5)$ as a linear combination of u_1 and u_2, where:

(a) $u_1 = (1, 2)$ and $u_2 = (3, 5)$;
(b) $u_1 = (3, -4)$ and $u_2 = (2, -3)$.

1.49. Write $v = \begin{bmatrix} 9 \\ -3 \\ 16 \end{bmatrix}$ as a linear combination of $u_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $u_2 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$.

1.50. Find k so that u and v are orthogonal, where:

(a) $u = (3, k, -2)$, $v = (6, -4, -3)$;
(b) $u = (5, k, -4, 2)$, $v = (1, -3, 2, 2k)$;
(c) $u = (1,\ 7,\ k+2,\ -2)$, $v = (3, k, -3, k)$.

Located Vectors, Hyperplanes, Lines in R^n

1.51. Find the vector v identified with the directed line segment $\overrightarrow{PQ}$ for the points:

(a) $P(2, 3, -7)$ and $Q(1, -6, -5)$ in $\mathbf{R}^3$;
(b) $P(1, -8, -4, 6)$ and $Q(3, -5, 2, -4)$ in $\mathbf{R}^4$.

1.52. Find an equation of the hyperplane H in $\mathbf{R}^4$ that:

(a) contains $P(1, 2, -3, 2)$ and is normal to $u = [2, 3, -5, 6]$;
(b) contains $P(3, -1, 2, 5)$ and is parallel to $2x_1 - 3x_2 + 5x_3 - 7x_4 = 4$.

1.53. Find a parametric representation of the line in $\mathbf{R}^4$ that:

(a) passes through the points $P(1, 2, 1, 2)$ and $Q(3, -5, 7, -9)$;
(b) passes through $P(1, 1, 3, 3)$ and is perpendicular to the hyperplane $2x_1 + 4x_2 + 6x_3 - 8x_4 = 5$.

Spatial Vectors (Vectors in R^3), ijk Notation

1.54. Given $u = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $v = 2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$, $w = 4\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$. Find:

(a) $2u - 3v$; (b) $3u + 4v - 2w$; (c) $u \cdot v$, $u \cdot w$, $v \cdot w$; (d) $\|u\|$, $\|v\|$, $\|w\|$.

1.55. Find the equation of the plane H:

(a) with normal $\mathbf{N} = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$ and containing the point $P(1, 2, -3)$;
(b) parallel to $4x + 3y - 2z = 11$ and containing the point $Q(2, -1, 3)$.

1.56. Find the (parametric) equation of the line L:

(a) through the point $P(2, 5, -3)$ and in the direction of $v = 4\mathbf{i} - 5\mathbf{j} + 7\mathbf{k}$;
(b) perpendicular to the plane $2x - 3y + 7z = 4$ and containing $P(1, -5, 7)$.

1.57. Consider the following curve C in $\mathbf{R}^3$ where $0 \le t \le 5$:

$$F(t) = t^3\mathbf{i} - t^2\mathbf{j} + (2t - 3)\mathbf{k}$$

(a) Find the point P on C corresponding to $t = 2$.
(b) Find the initial point Q and the terminal point Q'.
(c) Find the unit tangent vector $\mathbf{T}$ to the curve C when $t = 2$.

1.58. Consider a moving body B whose position at time t is given by $R(t) = t^2\mathbf{i} + t^3\mathbf{j} + 2t\mathbf{k}$. [Then $V(t) = dR(t)/dt$ and $A(t) = dV(t)/dt$ denote, respectively, the velocity and acceleration of B.] When $t = 1$, find for the body B:

(a) position; (b) velocity v; (c) speed s; (d) acceleration a.

1.59. Find a normal vector $\mathbf{N}$ and the tangent plane H to each surface at the given point:

(a) surface $x^2y + 3yz = 20$ and point $P(1, 3, 2)$;
(b) surface $x^2 + 3y^2 - 5z^2 = 160$ and point $P(3, -2, 1)$.

Cross Product

1.60. Evaluate the following determinants and negative of determinants of order two:

(a) $\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix}$, $\begin{vmatrix} 3 & -6 \\ 1 & -4 \end{vmatrix}$, $\begin{vmatrix} -4 & -2 \\ 7 & -3 \end{vmatrix}$

(b) $-\begin{vmatrix} 6 & 4 \\ 7 & 5 \end{vmatrix}$, $-\begin{vmatrix} 1 & -3 \\ 2 & 4 \end{vmatrix}$, $-\begin{vmatrix} 8 & -3 \\ -6 & -2 \end{vmatrix}$

1.61. Given $u = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $v = 2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$, $w = 4\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$, find:

(a) $u \times v$, (b) $u \times w$, (c) $v \times w$.

1.62. Given $u = [2, 1, 3]$, $v = [4, -1, 2]$, $w = [1, 1, 5]$, find:

(a) $u \times v$, (b) $u \times w$, (c) $v \times w$.

1.63. Find the volume V of the parallelopiped formed by the vectors u, v, w appearing in:

(a) Problem 1.61 (b) Problem 1.62.

1.64. Find a unit vector u orthogonal to:

(a) $v = [1, 2, 3]$ and $w = [1, -1, 2]$;
(b) $v = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $w = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$.

1.65. Prove the following properties of the cross product:

(a) $u \times v = -(v \times u)$
(b) $u \times u = 0$ for any vector u
(c) $(ku) \times v = k(u \times v) = u \times (kv)$
(d) $u \times (v + w) = (u \times v) + (u \times w)$
(e) $(v + w) \times u = (v \times u) + (w \times u)$
(f) $(u \times v) \times w = (u \cdot w)v - (v \cdot w)u$

Complex Numbers

1.66. Simplify:

(a) $(4 - 7i)(9 + 2i)$; (b) $(3 - 5i)^2$; (c) $\dfrac{1}{4 - 7i}$; (d) $\dfrac{9 + 2i}{3 - 5i}$; (e) $(1 - i)^3$.

1.67. Simplify: (a) $\dfrac{1}{2i}$; (b) $\dfrac{2 + 3i}{7 - 3i}$; (c) i^{15}, i^{25}, i^{34}; (d) $\left(\dfrac{1}{3 - i}\right)^2$.

1.68. Let $z = 2 - 5i$ and $w = 7 + 3i$. Find:

(a) $v + w$; (b) zw; (c) z/w; (d) $\bar{z}, \bar{w}$; (e) $|z|$, $|w|$.

1.69. Show that for complex numbers z and w:

(a) $\operatorname{Re} z = \frac{1}{2}(z + \bar{z})$, (b) $\operatorname{Im} z = \frac{1}{2}(z - \bar{z})$, (c) $zw = 0$ implies $z = 0$ or $w = 0$.

Vectors in C^n

1.70. Let $u = (1 + 7i, \ 2 - 6i)$ and $v = (5 - 2i, \ 3 - 4i)$. Find:

(a) $u + v$ (b) $(3 + i)u$ (c) $2iu + (4 + 7i)v$ (d) $u \cdot v$ (e) $\|u\|$ and $\|v\|$.

1.71. Prove: For any vectors u, v, w in $\mathbf{C}^n$:

(a) $(u+v)\cdot w = u\cdot w + v\cdot w$, (b) $w\cdot(u+v) = w\cdot u + w\cdot v$.

1.72. Prove that the norm in $\mathbf{C}^n$ satisfies the following laws:

$[N_1]$ For any vector u, $\|u\| \geq 0$; and $\|u\| = 0$ if and only if $u = 0$.
$[N_2]$ For any vector u and complex number z, $\|zu\| = |z|\|u\|$.
$[N_3]$ For any vectors u and v, $\|u+v\| \leq \|u\| + \|v\|$.

ANSWERS TO SUPPLEMENTARY PROBLEMS

1.41. (a) $(-3,-16,10)$; (b) (6,1,35); (c) $-3,-12,8$; (d) $\sqrt{21}, \sqrt{35}, \sqrt{14}$;
(e) $-3/\sqrt{21}\sqrt{35}$; (f) $\sqrt{62}$; (g) $-\frac{3}{35}(3,5,1) = (-\frac{9}{35}, -\frac{15}{35}, -\frac{3}{35})$

1.42. (Column vectors) (a) $(-1,7,-22)$; (b) $(-1,26,-29)$; (c) $-15,-27,34$;
(d) $\sqrt{26}, \sqrt{30}$; (e) $-15/(\sqrt{26}\sqrt{30})$; (f) $\sqrt{86}$; (g) $-\frac{15}{30}v = (-1, -\frac{1}{2}, -\frac{5}{2})$

1.43. (a) $(-7,-14,13,45,0)$; (b) $(20,-29,22,16,-23)$; (c) -6; (d) $\sqrt{90}, \sqrt{95}$;
(e) $-\frac{6}{95}v$; (f) $\sqrt{197}$

1.44. (a) $(5/\sqrt{74}, -7/\sqrt{74})$; (b) $(\frac{1}{5}, \frac{2}{5}, -\frac{2}{5}, \frac{4}{5})$; (c) $(6/\sqrt{133}, -4/\sqrt{133}, 9/\sqrt{133})$

1.45. (a) $3, 13, \sqrt{120}, 9$

1.46. (a) $x=3, y=5$; (b) $x=0, y=0$, and $x=-2, y=-4$

1.47. $x=-3$, $y=3$, $z=\frac{3}{2}$

1.48. (a) $v = 5u_1 - u_2$; (b) $v = 16u_1 - 23u_2$

1.49. $v = 3u_1 - u_2 + 2u_3$

1.50. (a) 6; (b) 3; (c) $\frac{3}{2}$

1.51. (a) $v = [-1,-9,2]$; (b) $[2,3,6,-10]$

1.52. (a) $2x_1 + 3x_2 - 5x_3 + 6x_4 = 35$; (b) $2x_1 - 3x_2 + 5x_3 - 7x_4 = -16$

1.53. (a) $[2t+1, -7t+2, 6t+1, -11t+2]$; (b) $[2t+1, 4t+1, 6t+3, -8t+3]$

1.54. (a) $-23\mathbf{j} + 13\mathbf{k}$; (b) $9\mathbf{i} - 6\mathbf{j} - 10\mathbf{k}$; (c) $-20,-12,37$; (d) $\sqrt{29}, \sqrt{38}, \sqrt{69}$

1.55. (a) $3x - 4y + 5z = -20$; (b) $4x + 3y - 2z = -1$

1.56. (a) $[4t+2, -5t+5, 7t-3]$; (b) $[2t+1, -3t-5, 7t+7]$

1.57. (a) $P = F(2) = 8\mathbf{i} - 4\mathbf{j} + \mathbf{k}$; (b) $Q = F(0) = -3\mathbf{k}$, $Q' = F(5) = 125\mathbf{i} - 25\mathbf{j} + 7\mathbf{k}$;
(c) $\mathbf{T} = (6\mathbf{i} - 2\mathbf{j} + \mathbf{k})/\sqrt{41}$

1.58. (a) $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$; (b) $2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$; (c) $\sqrt{17}$; (d) $2\mathbf{i} + 6\mathbf{j}$

1.59. (a) $\mathbf{N} = 6\mathbf{i} + 7\mathbf{j} + 9\mathbf{k}$, $6x + 7y + 9z = 45$; (b) $\mathbf{N} = 6\mathbf{i} - 12\mathbf{j} - 10\mathbf{k}$, $3x - 6y - 5z = 16$